MATH 20D Spring 2023 Lecture 10.

Conjugate roots, Free Mechanical Vibrations

Outline

Announcements

- Solutions to homework two are available in Canvas
- Midterm in Lecture this Wednesday.
  - Students are permitted **one double sided page of handwritten notes** as well as a **scientific calculator**.
  - Midterm review problem set is available in Canvas. Solutions available upon request in Zulip
  - The lecturer is confident that students who have studied the lecture examples and completed homeworks 1, 2 and 3 will succeed on the midterm.
  - In the interest of saving paper, students will **not** be provided with a table of standard integrals. The integrals on the exam do not require any techniques of integration beyond u substituition. Students are expected to know the antiderivatives of $\sin(x)$, $\cos(x)$, e^x and $1/x$.

Contents

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where $A = \sqrt{C_1^2 + C_2^2}$ and $\phi \in [0, 2\pi)$ satisfies $C_1 = A\sin(\phi)$ and $C_2 = A\cos(\phi)$.

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Example

The IVP $\frac{1}{8}y'' + 16y = 0$, $y(0) = 1/2$, $y'(0) = -\sqrt{2}$ admits the solution

$$y_{\text{sol}}(t) = \frac{1}{2}\cos(8\sqrt{2}t) - \frac{1}{8}\sin(8\sqrt{2}t).$$

Determine values of $A > 0$ and $\phi \in [0, 2\pi)$ so that $y_{\text{sol}}(t) = A\sin(8\sqrt{2}t + \phi)$

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*where $\alpha = -b/2m \leqslant 0$ and $\beta = \sqrt{4mk - b^2}/2m$. Hence if $b > 0$ then the mass oscillates with a decaying amplitude given the **damping factor** $Ae^{\alpha t}$.*

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